II European math tournament
Minsk, March 1–7, 2019

Senior Grand league. Math battle № 2

March, 4

1. Let's define $d(n)$ as the number of positive divisors of a positive integer n. Prove that for all positive integers n the following inequality holds:

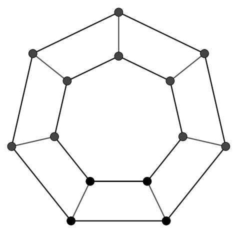
$$n + d(1) + d(2) + \ldots + d(n) \leqslant d(n+1) + d(n+2) + \ldots + d(2n).$$

2. A square $ABCD$ is given. Point P is selected inside $ABCD$ such that $AP = AB$ and $\angle CPD = 90°$. Prove that $DP = 2CP$.

3. There are a checkerboard 100×100, Nikolas and Alex are playing the following game. Nikolas begins and he can paint a cell in red on his turn, Alex can paint a cell in blue. They take turns and can't repaint a cell. At the end of the game, Nicholas finds a rectangle completely filled with red with the largest area, and Alex pays the amount of coins equal to the area of Nicholas's rectangle. What is the maximum profit that Nikolas can guarantee, no matter how Alex plays?

4. Kirill divided some number by 333 and found out that the sum of the quotient and the reminder is equal to 300. Nazar divided the same number by 777 and found out that the sum of the quotient and the reminder is equal to 300. Find the number.

5. Spider Andrew weaved a web, that you can see on the right. There are positive integers placed at the vertices. Equal numbers aren't connected. On each edge Andrew writes gcd (the greatest common divisor) of the numbers at the vertices. Is it possible that the sum of all numbers at the vertices is equal to the sum of all numbers on the edges?



6. For each number from 1 to 10 000 Pasha wrote down the product of all it's nonzero digits, and after that Kesha calculated the sum of all written numbers. What result did Kesha obtained?

7. There is a company of 100 girls, every girl have 100 candies. Each girl gave some of her candies to the other girls (girls couldn't give candies that they got from the others). At the end all girls have different amount of candies. Prove that at least one girl gave no less candies than she owned at the end.

8. Alice inserts plus after every even digit in a multidigit number and calculates the result. Bob inserts plus after every odd digit in the same multidigit number and calculates the result. For example, for number 2019 Alice gets $2+0+19 = 21$, and Bob gets $201+9 = 210$. Prove that there exist at least 100 6-digit numbers, for each of those Alice and Bob get the same result.

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International league. Math battle № 2

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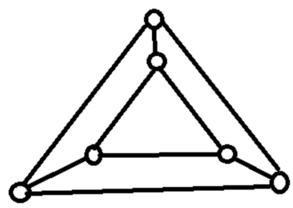
1. There are 25 irreducible positive fractions written on a piece of paper, the numerator of each fraction is less than the denominator. If you turn over all the fractions with denominators that are divisible by 3, then the product of all 25 fractions will be equal to 8,1. And what will be the product of all 25 fractions, if instead you turn over all the fractions with denominators not divisible 3?

2. Alice inserts plus after every even digit in a multidigit number and calculates the result. Bob inserts plus after every odd digit in the same multidigit number and calculates the result. For example, for number 2019 Alice gets $2+0+19 = 21$, and Bob gets $201+9 = 210$. Determine if whether exist 10 6-digit numbers consisting of distinct digits for each of those Alice and Bob get the same result.

3. There are 20 straight lines drawn on the plane, some are red, the others are blue. No three lines have a common point. It is known that the number of intersection points of lines of the same color is greater than the number of intersection points of lines of different colors. Prove that the numbers of red and blue lines cannot be equal.

4. In the circles (see the figure on the right) there are 6 different positive integers, and on each edge the greatest common divisor of the numbers at its ends is written. Can the sum of the numbers in the circles be equal to the sum of the numbers on the edges?



5. Is it possible to fill the table 3×3 with digits from 1 to 9 (using each digit once) so that the sum in the top row is three times more than the sum in the bottom row, and the sum in the right column is twice as much as in the left one?

6. 10 girls and 20 boys play a chess tournament. In each round, all the participants are divided into pairs of opponents. In the first three rounds, a girl played against a girl exactly 11 times. How many times in these three tours did a boy play against a boy?

7. In each cell of the 8×8 square stands a person: either a knight (always tells the truth) or a liar (he always lies). Neighbors are those who are in cells with a common side. Each person said the phrase “Among my neighbors there are more liars than knights”. What is the largest possible number of knights?

8. There are plastic rectangles and triangles, 77 pieces total (perhaps all 77 are of the same type). You can cut a single piece into two parts in a straight line once, and then you need to spread out all the pieces into three boxes. Prove that this can be done in such a way that the total number of angles in each box is the same (the triangle has three angles, the rectangle has four, the pentagon has five angles, etc., etc.).